

# Entropy Bound with Generalized Uncertainty Principle in General Dimensions

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(Dated: March 27, 2012)

## Abstract

In this letter, the entropy bound for local quantum field theories (LQFT) is studied in a class of models of the generalized uncertainty principle (GUP) which predicts a minimal length as a reflection of the quantum gravity effects. Both bosonic and fermionic fields confined in arbitrary spatial dimension  $d \geq 4$  ball  $\mathcal{B}^d$  are investigated. It is found that the GUP leads to the same scaling  $A_{d-2}^{(d-3)/(d-2)}$  correction to the entropy bound for bosons and fermions, although the coefficients of this correction are different for each case. Based on our calculation, we conclude that the GUP effects can become manifest at the short distance scale. Some further implications and speculations of our results are also discussed.

PACS numbers: 11.10.Kk, 04.70.Dy, 04.50.-h, 04.60.Bc, 03.67.-a

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## I. INTRODUCTION

Although a full description of quantum gravity is still missing, the existence of a minimal length appears as a common feature in various candidates of the quantum gravity, such as string theory[1, 2] and black hole physics[3]. Motivated by this progress, the idea of minimal length has attracted great interest in recent years.

Phenomenologically, the existence of a minimal length implies that a generalized uncertainty principle(GUP) should replace the ordinary one in the usual quantum mechanics. There are already many different proposals on the realization of this idea in the literature[4]. Recently, a new class of GUP models was proposed by Ali, Das, and Vagenas(ADV)[5] in order to incorporate the string theory, black hole physics and double special relativity (DSR). In their proposal, the minimal length can be characterized by the position and momentum operators satisfying generalized commutation relations[5]:

$$\begin{aligned} [x_i, x_j] &= [p_i, p_j] = 0, \\ [x_i, p_j] &= i\hbar[\delta_{ij} - \alpha(p\delta_{ij} + \frac{p_i p_j}{p}) + \alpha^2(p^2\delta_{ij} + 3p_i p_j)], \end{aligned} \quad (1)$$

which yields the generalized uncertainty principle to  $\alpha^2$  order

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle] \quad (2)$$

where  $\alpha = \alpha_0/M_{Pl}c = \alpha_0 l_{Pl}/\hbar$ , with  $M_{Pl}$  the Plank mass and  $l_{Pl}$  the Plank length. One can check that Eq.(1) given above is covariant under DSR transformation[6] which preserves a invariant energy scale. Thus it is natural to derive both a minimal measurable length and a maximal measurable momentum.i.e

$$\begin{aligned} \Delta x &\geq (\Delta x)_{min} \approx \alpha_0 l_{Pl}, \\ \Delta p &\leq (\Delta p)_{max} \approx \frac{M_{Pl}c}{\alpha_0}. \end{aligned} \quad (3)$$

Some phenomenological implications of above GUP have been studied in [7, 8]. An interesting effect is the modification of invariant weighted phase space volume. By rigorously solving the Liouville problem, the number of quantum states per momentum space volume is given by[8]

$$\frac{V}{(2\pi\hbar)^D} \frac{d^D p}{(1 - \alpha p)^{D+1}}. \quad (4)$$

On the other hand, it is generally believed that, at the fundamental level, the "holographic information bound" is an essential aspect of the nature of the quantum gravity[9, 10]. It states that the maximal entropy  $S_{max}$  in a volume  $R$  is bounded by the area of the boundary of the system  $A(\partial R)$ . A more stringent constraint can be obtained if one requires that the system can be described by some local quantum fields theory (LQFT). Cohen et.al. [11] proposed a entropy bound  $S_{max} \sim A^{\frac{3}{4}}$  for ordinary four-dimension spacetime LQFT by imposing the gravitational stability condition on the system. The mismatch of the entropy bound  $A^{\frac{3}{4}}$  and holographic bound  $A$  can be understood by the fact that the usual description of LQFT cannot work in the strongly gravitational circumstance.

Thus, it is interesting to consider the impact of GUP on the entropy bound, which can be regarded, at least partially, as a reflection of quantum gravity effect. With the method

presented in [12–14], this investigation[8, 15] has been done for the various GUP models, but their discussion was only restricted for bosonic fields in four dimension spacetime. In this letter, we present the entropy bound corrected by the model of GUP[5] whose commutation relations are given in Eq.(1). We do our calculation in the arbitrary dimension  $d \geq 4$  and both the bosonic and fermionic field are considered, generalizing the results given in [8]. It is found that this GUP model leads to the same scaling correction  $A_{d-2}^{(d-3)/(d-2)}$  to the ordinary entropy bound  $A_{d-2}^{(d-1)/d}$  for both the bosonic and fermionic systems, although the concrete coefficients before this correction are quite different. Based on our calculation, we conclude that the GUP effects can become manifest at the short distance scale. Some further implications and speculations of our results are also discussed.

In this paper, we will use the unit such that  $c = \hbar = G = k_B = 1$

## II. DERIVATION OF HOLOGRAPHIC ENTROPY BOUND FOR A REAL SCALAR BOSON IN GUP THEORY

In this section we give the detailed derivation of holographic entropy bound for the theory of the generalized uncertainty principle (GUP)[5] in arbitrary spacetime dimension  $d$ . First we consider the case of a free massless real scalar field  $\Phi(x)$ , which is confined in a  $(d-1)$ -dimensional spacelike ball  $\mathcal{B}_{d-1}^R$  with the radius  $R$ , as has been done in Ref. [13–15]. The modes of the field are then the solutions of the scalar wave equation  $\square\Phi = 0$  that vanish on the surface of the ball  $\partial\mathcal{B}_{d-1}^R$ , i.e. on the sphere  $\mathcal{S}_{d-2}^R$ . Using (4) and taking the continuous limit, we can perform the summation of quantities over the modes by the following replacement

$$\sum_{\mathbf{p}} \rightarrow \int \frac{d^{d-1}\mathbf{x} d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \frac{1}{(1-\alpha p)^d} = \frac{A_{d-2}(1)V_{d-1}(R)}{(2\pi)^{d-1}} \int \frac{dp p^{d-2}}{(1-\alpha p)^d}, \quad (5)$$

where  $V_{d-1}(R)$  is the  $(d-1)$ -dimensional volume of  $\mathcal{B}_{d-1}^R$  and  $A_{d-2}(R)$  the  $(d-2)$ -dimensional volume of the sphere  $\mathcal{S}_{d-2}^R$ , which can be explicitly given by:

$$V_n(R) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} R^n, \quad A_{n-1}(R) = \frac{2\pi^{n/2}}{\Gamma(n/2)} R^{n-1}. \quad (6)$$

In order to derive the holographic entropy bound for the GUP theory, we need to impose now two *ad hoc* restrictions on the admissible states in the bosonic Fock space. Firstly, we assume that the maximum energy of the allowed modes is given by the UV cutoff  $\Lambda$  of the local quantum field theory (LQFT). This condition effectively makes the total number of the quantized modes  $N$  finite, which is given by

$$\begin{aligned} N &= \sum_{\mathbf{p}} 1 \rightarrow \frac{A_{d-2}(1)V_{d-1}(R)}{(2\pi)^{d-1}} \int_0^\Lambda \frac{dp p^{d-2}}{(1-\alpha p)^d} \\ &\approx \frac{A_{d-2}(1)V_{d-1}(1)}{(2\pi)^{d-1}(d-1)} R^{d-1} \Lambda^{d-1} [1 + \alpha(d-1)\Lambda], \end{aligned} \quad (7)$$

where we assume that  $\alpha < \frac{1}{\Lambda}$  so that we can approximate the result by expanding the integrand to leading order in  $\alpha$ . Note that the total number of allowed modes increases due to the GUP correction.

Secondly, in order that the configuration of the states would not collapse to a black hole and the field states to be observable for outside world[9], we need to impose the maximum energy to a state in Fock space which should not be larger than the mass of the corresponding d-dimensional Schwarzschild black hole within the same size:

$$E_{BH}(R) = \frac{(d-2)A_{d-2}(R)}{16\pi R} = \eta R^{d-3}, \quad (8)$$

where, for later convenience, we have defined a dimensionless order 1 parameter  $\eta$ :

$$\eta \equiv \frac{(d-2)A_{d-2}(1)}{16\pi}. \quad (9)$$

The Fock states can be constructed by assigning the occupying number  $n_i$  to these  $N$  different modes,

$$|\Psi\rangle = |n(\mathbf{p}_1), n(\mathbf{p}_2), \dots, n(\mathbf{p}_N)\rangle \rightarrow |n_1, n_2, \dots, n_N\rangle. \quad (10)$$

The dimension of the Hilbert space is calculated by counting the possible distributions of the number occupancies  $\{n_i\}$ . The non-gravitational-collapse condition leads to finiteness of the Hilbert space:

$$E = \sum_{i=1}^N n_i p_i \leq E_{BH}(R) = \eta R^{d-3}. \quad (11)$$

Let us now consider a typical  $N$ -particle state with one particle occupying each mode ( $n_i = 1$ ), corresponding to the lowest energy state with  $N$  modes simultaneously excited. For this state, the total energy is

$$\begin{aligned} E_{tot} &= \sum_{i=1}^N p_i \rightarrow \frac{A_{d-2}(1)V_{d-1}(1)R^{d-1}}{(2\pi)^{d-1}} \int_0^\Lambda \frac{dp p^{d-1}}{(1-\alpha p)^d} \\ &\approx \mu R^{d-1} \Lambda^d \left[1 + \frac{\alpha d^2}{d+1} \Lambda\right], \end{aligned} \quad (12)$$

where we also expand the result to the leading order in the small parameter  $\alpha$  and absorb the various irrelevant dimensionless order 1 coefficients into a single one  $\mu$ :

$$\mu = \frac{A_{d-2}(1)V_{d-1}(1)}{(2\pi)^{d-1}d}. \quad (13)$$

By imposing the gravitational stability condition Eq.(11), one can arrive at the deformed UV-IR condition [11] due to the GUP effects

$$R^2 \Lambda^d \left(1 + \frac{\alpha d^2}{d+1} \Lambda\right) \leq \frac{\eta}{\mu}, \quad (14)$$

or, inversely, we can rewrite this condition in another form which is more convenient to use in the following:

$$\Lambda^d \leq \frac{\eta}{\mu} \frac{1}{R^2} \left[1 - \frac{\alpha d^2}{d+1} \left(\frac{\eta}{\mu}\right)^{1/d} R^{-2/d}\right]. \quad (15)$$

The entropy associated with the system is

$$S = - \sum_{i=1}^{W^b} \rho_i \ln \rho_i, \quad (16)$$

where  $\rho_i$  is the possibility distribution on the Hilbert state basis and  $W^b$  is the dimension of Hilbert space  $W^b = \dim \mathcal{H}$ . The maximum value of the expression can be reached by a uniform distribution  $\rho_i = \frac{1}{W}$  [12–15]. So the maximum entropy is given by

$$S_{max} = \ln W^b. \quad (17)$$

The bound of the dimension of the Hilbert space is studied in [12–14] and is determined by following formula:

$$W^b = \dim \mathcal{H} < \sum_{m=0}^N \frac{z^m}{(m!)^2} \leq \sum_{m=0}^{\infty} \frac{z^m}{(m!)^2} = I_0(2\sqrt{z}) \sim \frac{e^{2\sqrt{z}}}{\sqrt{4\pi\sqrt{z}}}, \quad (18)$$

where  $I_0(x)$  is the zeroth-order Bessel function of the second kind. And  $z$  is given by

$$\begin{aligned} z &= \sum_{i=1}^N \frac{E_{BH}(R)}{p} \rightarrow \frac{A_{d-2}(1)V_{d-1}(R)}{(2\pi)^{d-1}} \int_0^\Lambda \frac{E_{BH}(R)}{p} \frac{p^{d-2}dp}{(1-\alpha p)^d} \\ &\approx \mu\eta \frac{d}{d-2} R^{2(d-2)} \Lambda^{d-2} \left(1 + \frac{\alpha d(d-2)}{d-1} \Lambda\right). \end{aligned} \quad (19)$$

With the gravitational stability condition Eq.(11), we can further obtain the bound written in terms of the radius  $R$  which can also be regarded as the IR cutoff:

$$z \leq \mu\eta \frac{d}{d-2} \left(\frac{\eta}{\mu}\right)^{(d-2)/d} R^{2(d-1)(d-2)/d} \left[1 + 2\alpha \frac{d(d-2)}{(d+1)(d-1)} \left(\frac{\eta}{\mu}\right)^{1/d} R^{-2/d}\right]. \quad (20)$$

So the maximum entropy for the scalar field in the GUP background is

$$\begin{aligned} S_{max}^b &= \ln W^b \approx 2\sqrt{z} \\ &\leq 2\sqrt{\mu\eta \frac{d}{d-2}} \left(\frac{\eta}{\mu}\right)^{\frac{d-2}{2d}} R^{(d-1)(d-2)/d} \left[1 + \frac{\alpha d(d-2)}{(d+1)(d-1)} \left(\frac{\eta}{\mu}\right)^{1/d} R^{-2/d}\right] \\ &\propto A_{d-2}^{(d-1)/d} + \alpha \frac{d(d-2)}{(d+1)(d-1)} \left(\frac{\eta}{\mu}\right)^{1/d} A_{d-2}^{(d-3)/(d-2)}, \end{aligned} \quad (21)$$

where  $A_{d-2}$  is the area of the sphere enclosing the ball  $\mathcal{B}_{d-1}^R$ .

A simple but perhaps the most important example of this formula is the case when  $d=4$ , the holographic entropy bound for the scalar field is:

$$S_{max}^b(d=4) \leq A_2^{3/4} + \frac{16}{30} \alpha \left(\frac{\eta}{\mu}\right)^{1/4} A_2^{1/2} \quad (22)$$

which agrees with the result given in [8]

### III. DERIVATION OF HOLOGRAPHIC ENTROPY BOUND FOR A FERMION IN GUP THEORY

This section is devoted to the calculation of the holographic entropy bound for a fermionic degree of freedom. As the previous section, we also calculate the entropy bound for the fermion within a  $(d - 1)$ -dimensional ball. Due to the two constraints introduced in the previous section, the total number of allowed excited modes and the UV-IR relation are the same as those in the real scalar case, which is given in Eq.(7) and Eq.(15) respectively. However, because of the difference in the statistics of bosons from fermions, the dimension of the Hilbert space is not the same as before and we need some new insights for this case.

In fact, the counting of dimension of the Hilbert space is quite simple for the fermion case. By taking into account the Pauli exclusion principle, every mode can only have two states: empty and occupying. So for effective  $N$  excited modes, there are  $2^N$  states. Thus, the dimension of the Hilbert space is:

$$W^f = 2^N. \quad (23)$$

Following the arguments before, the distribution with the maximum entropy is the uniform one, so the maximum entropy is given by:

$$S_{max} = \ln W = \ln 2 \cdot N. \quad (24)$$

Due to the inequality Eq.(15) given by the gravitational stability, we can obtain a maximum bound for the allowed number of modes  $N$ , and in turn the bound for the entropy:

$$\begin{aligned} N &\approx \frac{A_{d-2}(1)V_{d-1}(1)}{(2\pi)^{d-1}(d-1)} R^{d-1} \Lambda^{d-1} [1 + \alpha(d-1)\Lambda] \\ &\leq \mu \frac{d}{d-1} \left(\frac{\eta}{\mu}\right)^{\frac{d-1}{d}} R^{(d-1)(d-2)/d} \left[1 + \alpha \frac{d-1}{d+1} \left(\frac{\eta}{\mu}\right)^{1/d} R^{-2/d}\right]. \end{aligned} \quad (25)$$

Therefore the holographic bound for a fermionic degree of freedom with the GUP effects is given by

$$S_{max}^f \leq A_{d-2}^{(d-1)/d} + \alpha \frac{d-1}{d+1} \left(\frac{\eta}{\mu}\right)^{1/d} A_{d-2}^{(d-3)/(d-2)}. \quad (26)$$

For the case of dimension 4 spacetime, we have

$$S_{max}^f \leq A_2^{3/4} + \frac{3}{5} \alpha \left(\frac{\eta}{\mu}\right)^{1/4} A_2^{1/2}. \quad (27)$$

Note that both the bosonic and fermionic fields share the same  $A_{d-2}^{(d-3)/(d-2)}$  scaling correction to the entropy bound due to the GUP, even though the coefficients before this correction are different for the two cases.

### IV. CONCLUSION AND DISCUSSION

In this paper, we study the impact of generalized uncertainty principle(GUP) proposed recently in [5] on the entropy bound for a local quantum field theory (LQFT) system in arbitrary dimensions, which can be regarded as the effects of quantum gravity. Both the bosonic and fermionic fields are considered. We find that the leading order  $\alpha$  correction to the

entropy bound goes as a positive one  $A_{d-2}^{(d-3)/(d-2)}$  for both cases, although the coefficients before this correction are quite different. According to our calculations above, we know that when the volume of  $\mathcal{S}^{d-2}$  is very large, the leading term  $A_{d-2}^{(d-1)/d}$  in Eqs.(21) and (26) dominates and the correction due to GUP effects can be ignorable. However, if volume of  $\mathcal{S}^{d-2}$  becomes smaller and reach the scale  $A_{d-2} \sim \alpha^{d(d-2)/2}$ , the GUP correction can compete with the ordinary leading term. When the volume becomes even smaller and approaches zero, the GUP effects begin to dominates the scaling behavior. Although the picture discussed above is only based upon our small  $\alpha$  approximation and higher order terms in the expansion of  $\alpha$  may change the scaling behavior dramatically, it is still sufficient to conclude that if the GUP really exists in nature, it can only manifest itself at the very short distance scale. This conclusion agrees with our usual straightforward intuition and the explanation that the GUP is the manifestation of the quantum gravity at some short distance scale.

Our calculation in this paper confirms the arguments in [8] that the holographic theories lose its good features due to the violation of continuous symmetry by the discrete space. Moreover, an interesting prospect comes up if one compares our results with a heuristic analysis[16] of black hole entropy corrected by the GUP model we considered here. The corrected entropy for Schwarzschild black hole in four dimension up to the first order is given by[16]

$$S \simeq A + \frac{\sqrt{\pi}}{2} \alpha \sqrt{A} \quad (28)$$

Although the leading order is different for black hole entropy bound and LQFT entropy bound, the scaling behavior of the first-order correction by the GUP is precisely the same, at least in four spacetime dimensions. Is it possible that the quantum gravity effects lead to some universal contributions for both holographic systems and LQFT systems? This needs more studies for the black hole entropy with the GUP, especially in general dimensions. The relevant research is being investigated.

### Acknowledgement

We would like to thank the 1<sup>st</sup> IAS-CERN School on Particle Physics and Cosmology and Implications for Technology at Nanyang Technological University for their hospitality and an stimulating environment. DH was supported in part by the National Science Foundation of China (NSFC) under Grant #No. 10821504, 10975170 and the Project of Knowledge Innovation Program (PKIP) of the Chinese Academy of Science. W. Wang was supported in part by funds from NSFC under Grant #No. 11075140 and Fundamental Research Funds for the Central University.

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